PSYCH 710

Random, Mixed, & Nested ANOVA

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fixed vs. random factors

- **fixed factor:** levels of factor would be the same in replication
- **random factor:** levels were selected randomly from large set & would vary across replications
  - introduces another source of variance that must be accounted for in our analyses

1-Way Random ANOVA

1-way random ANOVA

- alphas are assumed to be random variables selected from zero-mean, Normal distribution
- variance of scores is the sum of alpha & error variance

Full & Reduced Models:

$$Y_{ij} = \mu + \alpha_j + \epsilon_{ij}$$
$$Y_{ij} = \mu + \epsilon_{ij}$$

$$\mathrm{Var}(Y_{ij}) = \sigma^2_\alpha + \sigma^2_\epsilon$$

$$\mathrm{H0}: \sigma^2_\alpha = 0$$
$$\mathrm{H1}: \sigma^2_\alpha > 0$$

1-way random ANOVA

$$\text{H0}: \sigma_\alpha^2 = 0$$
$$\text{H1}: \sigma_\alpha^2 > 0$$

$$\xi\left[\frac{E_F}{df_F}\right] = \xi\left[\text{MS}_{WG}\right] = \sigma_\epsilon^2$$

When H0 is TRUE:

$$\xi\left[\text{MS}_{BG}\right] = \xi\left[\text{MS}_{WG}\right] = \sigma_\epsilon^2$$

$$\text{MS}_{BG} \approx \text{MS}_{WG}$$

$$F(a-1, N-a) = \frac{\text{MS}_{BG}}{\text{MS}_{WG}}$$

When H0 is FALSE:

$$\xi\left[\text{MS}_{BG}\right] = \xi\left[\frac{E_R - E_F}{df_R - df_F}\right] = \sigma_\epsilon^2 + n'\sigma_\alpha^2$$

$$\text{MS}_{BG} > \text{MS}_{WG}$$

$F \gg 1$ (for balanced designs, $n' = n$)

anova estimate of variance component

$$\text{Var}(Y_{ij}) = \sigma_\alpha^2 + \sigma_\epsilon^2$$

$\hat{\sigma}_\alpha^2 = (1/n) \times (\text{MS}_{BG} - \text{MS}_{WG})$ assumes equal n per group

(if $\hat{\sigma}_\alpha^2 < 0$ then set $\hat{\sigma}_\alpha^2$ to zero)

Ratio of variance components

$$\hat{\theta} = \frac{\hat{\sigma}_\alpha^2}{\hat{\sigma}_\epsilon^2} = \frac{\text{MS}_{\text{BG}} - m\text{MS}_{\text{WG}}}{mn\text{MS}_{\text{WG}}} \qquad m = \frac{a(n-1)}{a(n-1)-2}$$

$$\text{H0}: \frac{\sigma_\alpha^2}{\sigma_\epsilon^2} \le \theta_0$$

$$\text{H1}: \frac{\sigma_\alpha^2}{\sigma_\epsilon^2} > \theta_0$$

$$F_{obs} = \frac{\text{MS}_{\text{BG}}}{\text{MS}_{\text{WG}}}$$

reject H0 if $F_{obs} > (1 + n\theta_0)F_c$

$$F_c \ : \ p(F_c \,|\, \text{H0 is TRUE}) < \alpha \ \left(df_1 = (a-1);\ df_2 = a(n-1)\right)$$

strength of association

intraclass correlation ICC

$$\rho_I = \frac{\sigma_\alpha^2}{\sigma_\alpha^2 + \sigma_\epsilon^2}$$

$$\hat{\rho}_I = \frac{\text{MS}_{BG} - \text{MS}_{WG}}{\text{MS}_{BG} + (n-1)\text{MS}_{WG}} \qquad \hat{\rho}_I = \frac{F_{BG} - 1}{(n-1) + F_{BG}}$$

examples

1-way random anova

Random ANOVA & Variance Components (Dyestuff)

The Dyestuff data frame provides the yield of dyestuff (Naphthalene Black 12B) from 5 different preparations from each of 6 different batches of an intermediate product (H-acid). The Dyestuff2 data were generated data in the same structure but with a large residual variance relative to the batch variance.

```
> dye.aov.01 <- aov(Yield~Batch,data=Dyestuff)
> summary(dye.aov.01)
            Df Sum Sq Mean Sq F value Pr(>F)
Batch        5  56358   11272   4.598 0.0044 **
Residuals   24  58830    2451
> MS.batch <- 11272
> MS.error <- 2451
> var.comp.error <- MS.error
> n <- 5 # observations per cell
> ( var.comp.batch <- (MS.batch - MS.error)/n )
[1] 1764.2
> sqrt(var(Dyestuff$Yield)) # SD of sample yield
[1] 63.02367
> sqrt(var.comp.batch + var.comp.error) # estimated pop SD of yield
[1] 64.92457
```

Random ANOVA & Variance Components (Dyestuff)

The Dyestuff data frame provides the yield of dyestuff (Naphthalene Black 12B) from 5 different preparations from each of 6 different batches of an intermediate product (H-acid). The Dyestuff2 data were generated data in the same structure but with a large residual variance relative to the batch variance.

```
> MS.batch <- 11272
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> var.comp.error <- MS.error
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> ( var.comp.batch <- (MS.batch - MS.error)/n )
[1] 1764.2
> sqrt(var(Dyestuff$Yield)) # SD of sample yield
[1] 63.02367
> sqrt(var.comp.batch + var.comp.error) # estimated pop SD of yield
[1] 64.92457

> # unbiased estimate of variance ratio (theta)
> m <- 6 * (n-1) / (6 * (n-1) - 2)
> (11272 - m*2451) / (m*n*2451)
[1] 0.6431

> # association strength (ICC intraclass correlation):
> var.comp.batch / (var.comp.batch + var.comp.error)
[1] 0.4185
```

Random ANOVA & Variance Components (Dyestuff)

lmer in lmerTest package

```
> options(contrasts=c("contr.sum","contr.poly"))
> library(lmerTest)
> dye.lme <- lmer(Yield~(1|Batch),data=Dyestuff)
> ranova(dye.lme)
Model:
Yield ~ (1 | Batch)
            npar logLik AIC  LRT Df Pr(>Chisq)
<none>         3   -160 326
(1 | Batch)    2   -163 330 6.37  1      0.012 *

> VarCorr(dye.lme)
 Groups   Name        Std.Dev.        # variance components
 Batch    (Intercept) 42.0
 Residual             49.5

> ( ICC <- 42^2 / (42^2 + 49.5^2) ) # association strength
[1] 0.4186
```

Random ANOVA & Variance Components (Dyestuff)

anovaVCA in VCA package

```
> library(VCA)
> dye.VCA <- anovaVCA(Yield~Batch,Data=Dyestuff) # note capital D in Data
> print(dye.VCA) # print, not summary
Result Variance Component Analysis:
-----------------------------------
  Name  DF        SS      MS      VC      %Total    SD        CV[%]
1 total 15.101732                 4215.3  100       64.925342 4.250432
2 Batch 5         56357.5 11271.5 1764.05 41.848741 42.000595 2.74963
3 error 24        58830   2451.25 2451.25 58.151259 49.5101   3.24125
Mean: 1527.5 (N = 30)
Experimental Design: balanced  |  Method: ANOVA
```

intraclass correlation

Coefficient of Variation

```
> sd.batch <- 42
> sd.error <- 49.51
> # coefficients of variation
> ( CV.batch <- sd.batch/mean(Dyestuff$Yield) )
[1] 0.02749591
> ( CV.error <- sd.error/mean(Dyestuff$Yield) )
[1] 0.03241244
```

2-Way Mixed ANOVA

2-way mixed ANOVA

(A fixed; B random)

$$Y_{ijk} = \mu + \alpha_j + \beta_k + (\alpha\beta)_{jk} + \epsilon_{ijk}$$

- A is fixed: alphas are constrained (sum-to-zero)
- B is random: levels selected randomly
  - distributed normally with $\mu = 0$ and var $= \sigma^2_\beta$
- AxB interaction effects are random
  - distributed normally with $\mu = 0$ and var $= \sigma^2_{(\alpha\beta)}$
  - sum of interaction effects across levels of fixed factor is zero
  - sum of interaction effects within levels of fixed factor may not be zero

Effect of randomly sampling levels of a factor

- A = Therapy Mode (fixed)
  - all α's equal zero
- B = Clinical Trainee (random)
  - all β's equal zero
- AxB interaction effects (α,β) are not all zero
  - Across all possible levels, sum of (α,β)'s in each column & row = 0
  - When B is sampled, sum of (α,β)'s across levels of B may not be zero
  - AxB interaction leaks into main effect of A (i.e., the fixed factor)

TABLE 10.1
EXAMPLE OF THE EFFECTS OF AN INTERACTION IN THE POPULATION BETWEEN A RANDOM FACTOR AND A FIXED FACTOR

I. Population Means for Three Therapy Modes and for the Entire Population of Trainees

	Clinical Trainee																		
Therapy Mode	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*	*k*	*l*	*m*	*n*	*o*	*p*	*q*	*r*	*Mean*
Psychodynamic	7	6	5	7	6	5	4	4	4	1	2	3	4	4	4	1	2	3	4
Behavioral	4	4	4	1	2	3	7	6	5	7	6	5	1	2	3	4	4	4	4
Rogerian	1	2	3	4	4	4	1	2	3	4	4	4	7	6	5	7	6	5	4
Mean	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4

II. Population Means for Three Therapy Modes and for a Sample of Trainees

	Clinical Trainee			
Therapy Mode	*g*	*k*	*r*	*Mean*
Psychodynamic	4	2	3	3.00
Behavioral	7	6	4	5.67
Rogerian	1	4	5	3.33
Mean	4	4	4	4.00

Maxwell, Delaney & Kelly (2018)

Expected values of Mean Squares

Source	$\xi(MS)$ A fixed; B random
A	$\sigma_\epsilon^2 + n\sigma_{(\alpha\beta)}^2 + bn\sum_{j=1}^{a}\alpha_j^2/(a-1)$
B	$\sigma_\epsilon^2 + an\sigma_\beta^2$
$A \times B$	$\sigma_\epsilon^2 + n\sigma_{(\alpha\beta)}^2$
Residuals	σ_ϵ^2

Evaluating main effect of A

Source	$\xi(MS)$ A fixed; B random
A	$\sigma_\epsilon^2 + n\sigma_{(\alpha\beta)}^2 + bn\sum_{j=1}^{a}\alpha_j^2/(a-1)$
B	$\sigma_\epsilon^2 + an\sigma_\beta^2$
$A \times B$	$\sigma_\epsilon^2 + n\sigma_{(\alpha\beta)}^2$
Residuals	σ_ϵ^2

$$\begin{aligned} \text{H0}: \alpha_j &= 0 \text{ for all levels } j \\ \text{H1}: \alpha_j &\neq 0 \text{ for at least one level } j \end{aligned}$$

When H0 is TRUE:

$$\text{MS}_A \approx \text{MS}_{A\times B}$$

Evaluating main effect of B

Source	$\xi(MS)$ A fixed; B random
A	$\sigma_\epsilon^2 + n\sigma_{(\alpha\beta)}^2 + bn\sum_{j=1}^{a}\alpha_j^2/(a-1)$
B	$\sigma_\epsilon^2 + an\sigma_\beta^2$
$A \times B$	$\sigma_\epsilon^2 + n\sigma_{(\alpha\beta)}^2$
Residuals	σ_ϵ^2

$$\begin{aligned} \text{H0}: \sigma_\beta^2 &= 0 \\ \text{H1}: \sigma_\beta^2 &> 0 \end{aligned}$$

When H0 is TRUE:

$$\text{MS}_B \approx \text{MS}_{Residuals}$$

Evaluating A x B interaction

Source	$\xi(MS)$ A fixed; B random
A	$\sigma_\epsilon^2 + n\sigma_{(\alpha\beta)}^2 + bn\sum_{j=1}^{a}\alpha_j^2/(a-1)$
B	$\sigma_\epsilon^2 + an\sigma_\beta^2$
$A \times B$	$\sigma_\epsilon^2 + n\sigma_{(\alpha\beta)}^2$
Residuals	σ_ϵ^2

$$\begin{aligned} \text{H0}: \sigma_{(\alpha\beta)}^2 &= 0 \\ \text{H1}: \sigma_{(\alpha\beta)}^2 &> 0 \end{aligned}$$

When H0 is TRUE:

$$\text{MS}_{A\times B} \approx \text{MS}_{Residuals}$$

strength of association

mixed 2-way ANOVA

$$\omega^2_{A,partial} = \frac{\sum_{j=1}^{a}(\alpha_j^2/a)}{\sigma_\epsilon^2 + \sum_{j=1}^{a}(\alpha_j^2/a)}$$

$$\hat{\omega}^2_{A,partial} = \frac{(a-1)(F_A-1)}{(a-1)(F_A-1)+nab}$$

$$\rho_{I:B,partial} = \frac{\sigma_\beta^2}{\sigma_\beta^2 + \sigma_\epsilon^2}$$

$$\hat{\sigma}_\beta^2 = \frac{\text{MS}_B - \text{MS}_R}{na}$$

$$\hat{\sigma}_{(\alpha\beta)}^2 = \frac{\text{MS}_{A\times B} - \text{MS}_R}{n}$$

$$\rho_{I:AB,partial} = \frac{\sigma_{(\alpha\beta)}^2}{\sigma_{(\alpha\beta)}^2 + \sigma_\epsilon^2}$$

$$\hat{\sigma}_\epsilon^2 = \text{MS}_R$$

Example (ACT Experiment)

mixed-effect ANOVA

- study compared 2 programs to prepare for ACT
- crossed-factorial design:
  - schools (random) x study program (fixed)

```
> mw.act <- read.table("chapter_10_table_5.dat")
> names(mw.act)<-c("study","school","score")
> # study: fixed; school: random
> mw.act$study<-factor(mw.act$study,labels=c("computer","standard"))
> mw.act$school<-factor(mw.act$school,labels="s")
> with(mw.act,tapply(score,list(study,school),length))
         s1 s2 s3 s4
computer  5  5  5  5
standard  5  5  5  5
```

Example (ACT Experiment)

mixed-effect ANOVA

```
> options(contrasts=c("contr.sum","contr.poly"))
> summary(aov(score~study+school+study:school,data=mw.act))
             Df Sum Sq Mean Sq F value   Pr(>F)
study         1    360   360.0  19.931 9.35e-05 ***
school        3    100    33.3   1.845    0.159
study:school  3     80    26.7   1.476    0.240
Residuals    32    578    18.1
```

```
> MS.study <- 360
> MS.AxB <- 26.67
> df.AxB <- 3
> ( F.study <- MS.study/MS.AxB )
[1] 13.49831
> ( p.study <- 1-pf(F.study,1,df.AxB))
[1] 0.03490257
```

recalculate F & p for fixed effect

```
> ( var.comp.school <- (33.33-26.67)/(2*5))
[1] 0.666
> ( var.comp.schoolxstudy <- (26.67-18.06)/5)
[1] 1.722
```

- very small variance components
- this result may be interesting/important

Example (ACT Experiment)

mixed-effect ANOVA with lmer

```
> require(lmerTest)
> act.lme.01 <- lmer(score ~ study + (1|school) + (1|study:school),data=mw.act)
```

```
> anova(act.lme.01)
Type III Analysis of Variance Table with Satterthwaite's method
      Sum Sq Mean Sq NumDF DenDF F value Pr(>F)
study 243.84  243.84     1     3    13.5 0.0349 *
```

Fixed Effect ANOVA

Example (ACT Experiment)

mixed-effect ANOVA with lmer

```
> require(lmerTest)
> act.lme.01 <- lmer(score ~ study + (1|school) + (1|study:school),data=mw.act)
```

```
> VarCorr(school.lme) # var components as std dev
 Groups       Name        Std.Dev.
 study:school (Intercept) 1.312
 school       (Intercept) 0.817
 Residual                 4.250
```

Variance Components

```
> ranova(school.lme) # significance tests
ANOVA-like table for random-effects: Single term deletions
Model:
score ~ study + (1 | school) + (1 | study:school)
                   npar logLik AIC    LRT Df Pr(>Chisq)
<none>                5   -114 238
(1 | school)          4   -114 236 0.0373  1       0.85
(1 | study:school)    4   -114 236 0.2319  1       0.63
```

dropping random effects does NOT change fit significantly

Example (ACT Experiment)

mixed-effect ANOVA with lmer

```
> # association strength for random effects
> library(performance)
> icc(school.lme,by_group=T)
# ICC by Group
Group         |   ICC
--------------------
study:school  | 0.084
school        | 0.033
```

```
> # association strength for fixed effects
> library(effectsize)
> omega_squared(school.lme)
# Effect Size for ANOVA (Type III)

Parameter | Omega2 (partial) |       95% CI
-------------------------------------------
study     |             0.71 | [0.00, 1.00]
```

Example (ACT Experiment)

mixed-effect ANOVA [variance components calculated with anovaMM]

```
> require(VCA)
> act.MM <- anovaMM(score~study + (school) + (study:school),Data=mw.act)
> print(act.MM,digits=3)
ANOVA-Type Estimation of Mixed Model:
--------------------------------------
       [Fixed Effects]
          int studycomputer studystandard
           27            -6             0
       [Variance Components]
  Name          DF     SS  MS     VC     %Total SD    CV[%]
1 total         33.193            20.45  100    4.522 18.842
2 school        3      100 33.333 0.667  3.26   0.816 3.402
3 study:school  3      80  26.667 1.721  8.415  1.312 5.466
4 error         32     578 18.062 18.062 88.325 4.25  17.708

Mean: 24 (N = 40)
Experimental Design: balanced  |  Method: ANOVA
```

ICC

2-Way Random ANOVA

2-way random ANOVA

(A & B random)

$$Y_{ijk} = \mu + \alpha_j + \beta_k + (\alpha\beta)_{jk} + \epsilon_{ijk}$$

- A & B are random:
  - distributed normally with μ=0 and var=σ^2_β
- interaction effects are random
  - distributed normally with μ=0 and var=$\sigma^2_{(\alpha\beta)}$
- main effects evaluated by comparing MS_A & MS_B to MS_{AxB}
- interaction evaluated by comparing MS_{AxB} to $MS_{residuals}$

Source	$\xi(MS)$ A & B random
A	$\sigma^2_\epsilon + n\sigma^2_{(\alpha\beta)} + bn\sigma^2_\alpha$
B	$\sigma^2_\epsilon + n\sigma^2_{(\alpha\beta)} + an\sigma^2_\beta$
$A \times B$	$\sigma^2_\epsilon + n\sigma^2_{(\alpha\beta)}$
Residuals	σ^2_ϵ

$$H0: \ \sigma^2_\alpha = 0$$
$$H1: \ \sigma^2_\alpha > 0$$

2-way random ANOVA

(A & B random)

$$Y_{ijk} = \mu + \alpha_j + \beta_k + (\alpha\beta)_{jk} + \epsilon_{ijk}$$

- A & B are random:
  - distributed normally with μ=0 and var=σ^2_β
- interaction effects are random
  - distributed normally with μ=0 and var=$\sigma^2_{(\alpha\beta)}$
- main effects evaluated by comparing MS_A & MS_B to MS_{AxB}
- interaction evaluated by comparing MS_{AxB} to $MS_{residuals}$

Source	$\xi(MS)$ A & B random
A	$\sigma^2_\epsilon + n\sigma^2_{(\alpha\beta)} + bn\sigma^2_\alpha$
B	$\sigma^2_\epsilon + n\sigma^2_{(\alpha\beta)} + an\sigma^2_\beta$
$A \times B$	$\sigma^2_\epsilon + n\sigma^2_{(\alpha\beta)}$
Residuals	σ^2_ϵ

$$H0: \ \sigma^2_\beta = 0$$
$$H1: \ \sigma^2_\beta > 0$$

2-way random ANOVA

(A & B random)

$$Y_{ijk} = \mu + \alpha_j + \beta_k + (\alpha\beta)_{jk} + \epsilon_{ijk}$$

- A & B are random:
  - distributed normally with μ=0 and var=σ^2_β
- interaction effects are random
  - distributed normally with μ=0 and var=$\sigma^2_{(\alpha\beta)}$
- main effects evaluated by comparing MS_A & MS_B to MS_{AxB}
- interaction evaluated by comparing MS_{AxB} to $MS_{residuals}$

Source	$\xi(MS)$ A & B random
A	$\sigma^2_\epsilon + n\sigma^2_{(\alpha\beta)} + bn\sigma^2_\alpha$
B	$\sigma^2_\epsilon + n\sigma^2_{(\alpha\beta)} + an\sigma^2_\beta$
$A \times B$	$\sigma^2_\epsilon + n\sigma^2_{(\alpha\beta)}$
Residuals	σ^2_ϵ

$$H0: \ \sigma^2_{(\alpha\beta)} = 0$$
$$H1: \ \sigma^2_{(\alpha\beta)} > 0$$

strength of association

random 2-way ANOVA

$$\rho_{I:A,partial} = \frac{\sigma^2_\alpha}{\sigma^2_\alpha + \sigma^2_\epsilon}$$

$$\rho_{I:B,partial} = \frac{\sigma^2_\beta}{\sigma^2_\beta + \sigma^2_\epsilon}$$

$$\rho_{I:AB,partial} = \frac{\sigma^2_{(\alpha\beta)}}{\sigma^2_{(\alpha\beta)} + \sigma^2_\epsilon}$$

$$\hat{\sigma}^2_\alpha = \frac{\mathrm{MS}_A - \mathrm{MS}_{A\times B}}{nb}$$

$$\hat{\sigma}^2_\beta = \frac{\mathrm{MS}_B - \mathrm{MS}_{A\times B}}{na}$$

$$\hat{\sigma}^2_{(\alpha\beta)} = \frac{\mathrm{MS}_{A\times B} - \mathrm{MS}_R}{n}$$

$$\hat{\sigma}^2_\epsilon = \mathrm{MS}_R$$

Example (ACT Experiment)

two-way random ANOVA

```
> options(contrasts=c("contr.sum","contr.poly"))
> act.aov.01 <- aov(score~study+school+study:school,data=mw.act)
> summary(act.aov.01)
             Df Sum Sq Mean Sq F value  Pr(>F)
study         1    360     360   19.93 9.4e-05 ***
school        3    100      33    1.85    0.16
study:school  3     80      27    1.48    0.24
Residuals    32    578      18
> # assume both factors are random:
> MS.axb <- 26.7
> (F.study <- 360/MS.axb)
[1] 13.48
> (p.study <- 1-pf(F.study,1,3))
[1] 0.03495
> (F.school <- 33.3/MS.axb)
[1] 1.247
> (p.school <- 1-pf(F.school,3,3))
[1] 0.4301
```

recalculate F & p for both main effects

Example (ACT Experiment)

ANOVA variance components

$$\hat{\sigma}^2_\alpha = \frac{\mathrm{MS}_A - \mathrm{MS}_{A\times B}}{nb}$$

$$\hat{\sigma}^2_\beta = \frac{\mathrm{MS}_B - \mathrm{MS}_{A\times B}}{na}$$

$$\hat{\sigma}^2_{(\alpha\beta)} = \frac{\mathrm{MS}_{A\times B} - \mathrm{MS}_R}{n}$$

$$\hat{\sigma}^2_e = \mathrm{MS}_R$$

```
> xtabs(~school+study,data=mw.act)
               study
school computer standard
    s1        5        5
    s2        5        5
    s3        5        5
    s4        5        5
> # variance components
> a <- 4 # levels of schools
> b <- 2 # levels of study
> n <- 5 # per group
> ( var.comp.school <- (33-27)/(n*b))
[1] 0.6
> ( var.comp.study <- (360-27)/(n*a))
[1] 16.65
> ( var.comp.study_x_school <- (27-18)/n )
[1] 1.8
```

- 2 of these variance components are small

Example (ACT Experiment)

ANOVA variance components

```
> library(VCA)
> anovaVCA(score ~ 1 + study*school,Data=mw.act)

Result Variance Component Analysis:
-----------------------------------

  Name         DF       SS  MS        VC        %Total    SD       CV[%]
1 total        4.114573             37.116667 100       6.092345 25.384771
2 study        1        360 360       16.666667 44.903458 4.082483 17.010345
3 school       3        100 33.333333 0.666667  1.796138  0.816497 3.402069
4 study:school 3        80  26.666667 1.720833  4.636282  1.311805 5.465856
5 error        32       578 18.0625   18.0625   48.664122 4.25     17.708333

Mean: 24 (N = 40)

Experimental Design: balanced  |  Method: ANOVA
```

Example (ACT Experiment)

two-way random factorial (with lmer)

```
> # with lmer
> library(lmerTest)
> options(contrasts=c("contr.sum","contr.poly"))
> act.lme <- lmer(score ~ 1 + (1|study) + (1|school) + (1|study:school),data=mw.act)
> print(VarCorr(act.lme),comp="Variance")
 Groups       Name        Variance
 study:school (Intercept)  1.721
 school       (Intercept)  0.667
 study        (Intercept) 16.667
 Residual                 18.062
> ranova(act.lme)
boundary (singular) fit: see ?isSingular
Model:
score ~ (1 | study) + (1 | school) + (1 | study:school)
                     npar logLik AIC  LRT Df Pr(>Chisq)
<none>                  5   -117 243
(1 | study)             4   -119 245 4.16  1      0.041 *
(1 | school)            4   -117 241 0.04  1      0.847
(1 | study:school)      4   -117 242 0.23  1      0.630
```

one of the random effects might be zero

Nested Factors

Nested Experimental Design

- Factor B is nested within factor A
  - if each level of B occurs within only one level of A
- Nesting often is a result of sampling strategy

Table 4: Example of a nested design.

	b_1	b_2	b_3	b_4	b_5	b_6
a_1	6	6	6	x	x	x
a_2	x	x	x	6	6	6

Example of nested sampling strategy

- experimenter estimates calcium concentration from turnip leaves
- randomly selects 4 turnip plants
  - within each plant, randomly samples 3 leaves
    - within each leaf, randomly samples 3 locations
      - measures calcium in each location
- final measure depends on variation among plants
  - variation among leaves within plant
    - variation among locations within leaves

Evaluating effects of A & B

$$Y_{ijk} = \mu + \alpha_j + \beta_{k/j} + \epsilon_{ijk}$$

b is the number of levels of B in each level of A

Table 5: Expected mean squares for nested designs (B nested within A).

Source	$\xi(MS)$ A fixed; B random	$\xi(MS)$ A & B random	df
A	$\sigma_e^2 + n\sigma_\beta^2 + bn\sum_{j=1}^{a}\alpha_j/(a-1)$	$\sigma_e^2 + n\sigma_\beta^2 + bn\sigma_\alpha^2$	$a-1$
B/A	$\sigma_e^2 + n\sigma_\beta^2$	$\sigma_e^2 + n\sigma_\beta^2$	$\sum_{j=1}^{a}(b-1) = a(b-1)$
Residuals	σ_e^2	σ_e^2	$ab(n-1)$

$$F_A = \frac{\text{MS}_A}{\text{MS}_{\text{B/A}}} \qquad F_{\text{B/A}} = \frac{\text{MS}_{\text{B/A}}}{\text{MS}_{\text{Residuals}}}$$

Variance components

B nested within A

$$\hat{\sigma}_\beta^2 = \frac{MS_{B/A} - MS_{Residuals}}{n} \qquad \hat{\sigma}_\alpha^2 = \frac{MS_A - MS_{B/A}}{bn}$$

n : observations per cell
b : number of levels of B within each level of A

Example: turnips

```r
load(url("http://pnb.mcmaster.ca/bennett/psy710/datasets/turnips.rda"))
xtabs(~plant+leaf,data=turnips)

##       leaf
## plant L1 L2 L3 L4 L5 L6 L7 L8 L9 L10 L11 L12
##    p1  2  2  2  0  0  0  0  0  0   0   0   0
##    p2  0  0  0  2  2  2  0  0  0   0   0   0
##    p3  0  0  0  0  0  0  2  2  2   0   0   0
##    p4  0  0  0  0  0  0  0  0  0   2   2   2
```

```
> turnips
   plant leaf spot calcium
1     p1   L1   s1    3.28
2     p1   L1   s2    3.09
3     p1   L2   s3    3.52
4     p1   L2   s4    3.48
5     p1   L3   s5    2.88
6     p1   L3   s6    2.80
7     p2   L4   s7    2.46
8     p2   L4   s8    2.44
9     p2   L5   s9    1.87
10    p2   L5  s10    1.92
11    p2   L6  s11    2.19
12    p2   L6  s12    2.19
13    p3   L7  s13    2.77
14    p3   L7  s14    2.66
15    p3   L8  s15    3.74
16    p3   L8  s16    3.44
17    p3   L9  s17    2.55
18    p3   L9  s18    2.55
19    p4  L10  s19    3.78
20    p4  L10  s20    3.87
21    p4  L11  s21    4.07
22    p4  L11  s22    4.12
23    p4  L12  s23    3.31
24    p4  L12  s24    3.31
```

- Each plant, each leaf, and each spot is given a unique identifier
- Necessary if you want lmer to figure out that leaf is NESTED in plant and spot is NESTED in leaf

Example: turnips

using aov()

```
> summary(aov(calcium~plant+leaf,data=turnips) )
            Df Sum Sq Mean Sq F value  Pr(>F)
plant        3   7.56   2.520   378.7 3.8e-12 ***
leaf         8   2.63   0.329    49.4 5.1e-08 ***
Residuals   12   0.08   0.007

> (F.plant <- 2.5201/.3288) # denominator from leaf
[1] 7.665
> (p.plant <- 1-pf(F.plant,3,8) )
[1] 0.009727
```

$$F_A = \frac{MS_A}{MS_{B/A}} \qquad F_{B/A} = \frac{MS_{B/A}}{MS_{Residuals}}$$

Example: turnips

using aov()

```
> n <- 2
> a <- 4
> b <- 3

> # variance components (anova estimates):
> var.comp.error <- .0067
> ( var.comp.leaf <- (.3288-.0067) / n )
[1] 0.161
> ( var.comp.plant <- (2.5201 - .3288) / (b*n) )
[1] 0.3652

> # partial ICC:
> (part.icc.leaf <- var.comp.leaf / (var.comp.leaf + var.comp.error) )
[1] 0.9601
> (part.icc.plant <- var.comp.plant / (var.comp.plant + var.comp.error) )
[1] 0.982
```

Example: turnips

analysis with lmer

```
> library(lmerTest)
> turnip.lme.01 <- lmer(calcium ~ 1 + (1|plant) + (1|leaf),data=turnips)
> anova(turnip.lme.01) # nothing here; no fixed effects
Type III Analysis of Variance Table with Satterthwaite's method
     Sum Sq Mean Sq NumDF DenDF F value Pr(>F)


> ranova(turnip.lme.01) # chi-square tests on random effects
ANOVA-like table for random-effects: Single term deletions

Model:
calcium ~ (1 | plant) + (1 | leaf)
             npar logLik  AIC   LRT Df Pr(>Chisq)
<none>          4  -1.09 10.2
(1 | plant)     3  -3.73 13.5  5.29  1      0.022 *
(1 | leaf)      3 -15.62 37.2 29.07  1      7e-08 ***
```

Example: turnips

variance components & association strength

Variance Components

```
vc <- VarCorr(turnip.lme.01)
print(vc,comp="Variance")

##   Groups   Name        Variance
##   leaf     (Intercept) 0.16117
##   plant    (Intercept) 0.36460
##   Residual             0.00665
```

variation within a leaf is VERY small
compared to variation across leaves & plants

Intraclass Correlation

```
library(performance)
icc(turnip.lme.01,by_group=T)

## # ICC by Group
##
## Group |   ICC
## -------------
## leaf  | 0.303
## plant | 0.685
```

ICC (not partial-ICC)

$$\hat{\rho}^2_{plant} = \frac{\hat{\sigma}^2_{plant}}{\hat{\sigma}^2_{plant} + \hat{\sigma}^2_{leaf} + \hat{\sigma}^2_{error}}$$